Lecture 15

<u>Integration of $\mathbb{C}$-valued fcns.</u> $(X, \mathcal{M}, \mu)$ given:

First, if $f: X \xrightarrow{\text{measurable}} \mathbb{R}$ is decomposed $f = f^+ - f^-$, then $f^+, f^- \in L^+$, so $\int f^+$, $\int f^-$ are defined.

If $\int f^+ < \infty$ <u>or</u> $\int f^- < \infty$, may define

$$\int f = \int f^+ - \int f^-.$$

<u>Def'n</u>. ① If $\int f^+ < \infty$ <u>and</u> $\int f^- < \infty$ (or equivalently $\int |f| < \infty$), then we say f is integrable.

② If $f: X \to \mathbb{C}$, we may decompose $f = u + iv$, where $u, v: X \to \mathbb{R}$. We say f is integrable if u, v are integrable and define

$$\int f = \int u + i \int v.$$

We shall use notation $L^1(X,\mu) = L^1(\mu) = L^1$ for space of $\mathbb{C}$-valued integrable fcns.

__Obs.__ $L^1(X,\mu)$ is a vector space / $\mathbb{C}$ and $\int : L^1 \to \mathbb{C}$ is linear.

__Prop__ Let $f,g \in L^1(X,\mu)$.

(i) $|\int f| \le \int |f|$

(ii) $\{x : f(x) \ne 0\}$ is σ-finite.

(iii) $\int_E f = \int_E g$, $\forall E \in \mathcal{M} \iff f = g$ μ-a.e.

__Pf.__ (i) DIY.

(ii) Part of HW.

(iii) Let $h = f - g$. Since $h = 0 \iff |h| = 0$, we see $h = 0 \Rightarrow \int_X |h| = 0$ by Folland Prop 2.16 and hence $\int_E |h| = 0$, $\forall E \in \mathcal{M}$ by monotonicity. $\int_E h\,d\mu = 0$, $\forall E$ then follows by (i). Thus, "$\Leftarrow$" is proved.

To establish converse, suppose $h \neq 0$ on E w/ $\mu(E) > 0$. Writing $h = u + iv = u^+ - u^- + i(v^+ - v^-)$ we note at least on of u^+, u^-, v^+, v^- must be > 0 on E. Suppose $u^+ > 0$ on E. Then, $u^- = 0$ on E and $\int_E u > 0$ by Prop 2.16

$\Rightarrow \mathrm{Re} \int_E h > 0 \Rightarrow \int_E h \neq 0$, proving "$\Rightarrow$". ∎

At this point, let us modify our def. of $L^1(X, \mu)$. For $f, g : X \to \mathbb{C}$ integrable, we say $f \sim g$ if $f = g$ μ-a.e. and let
$$L^1(X, \mu) = \{ \underbrace{[f]}_{\text{eq. classes}} : f : X \to \mathbb{C} \text{ integrable} \}.$$

Then L^1 is still a vector space / $\mathbb{C}$ and $\int : L^1 \to \mathbb{C}$ is a well-defined$^{(*)}$ linear functional. We make L^1 into a normed (metric) space by

$$\|f\|_{L^1} = \int_X |f| \, d\mu .$$

(*) by Prop 7.

Thus, the metric is $d(f,g) = \|f-g\|_{L^1}$.

Note: (1) Even though elements of L^1 are eq. classes, we speak of "functions" in L^1, referring to some repr. of the eq. class.

(2) This avoids the issue that may arise if μ is not complete: If $\{f_n\}$ is a seq. of meas. fcns s.t. $f_n \to f$ μ-a.e., f need not be meas. but if $f_n \to f$ on E w/ $\mu(E^c)=0$, then $\chi_E f_n \to \chi_E f$ everywhere, so $\chi_E f$ is meas. and $\chi_E f = f$ μ-a.e.

A fundamental convergence result:

<u>Dominated Convergence Thm</u>. Let $\{f_n\}_{n=1}^{\infty} \subseteq L^1(X,\mu)$ s.t. $f_n \to f$ μ-a.e. and $\exists\, g \in L^1$ s.t. $|f_n| \leq g$ μ-a.e.

Then, $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.